Section 14.1: Multivariable functions 9/27/21

Definition: A multivariable function (of n variables with real values) is a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$

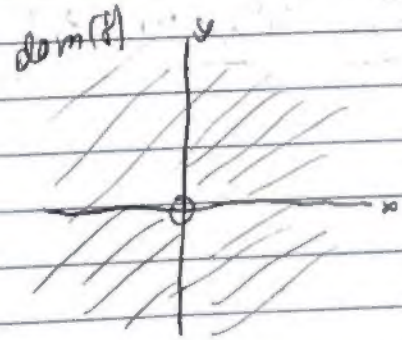
function's name   Domain   number of variables

$\text{dom}(f)$ = domain of f
$\text{ran}(f) = \{ f(\vec{x}) : \vec{x} \in \text{dom}(f) \}$

NB: Often we won't explicitly state the domain of a function given formulaically. We'll use "the natural domain" in that case, ie. the set of all inputs w/ defined outputs given by the formula

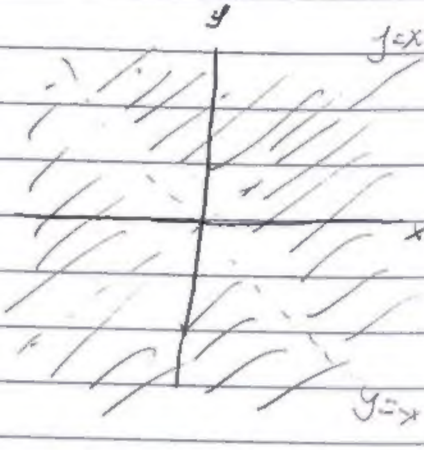
Ex: $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$

$$\begin{aligned} \text{dom}(f) &= \{ (x,y) \in \mathbb{R}^2 : \tfrac{x^2-y^2}{x^2+y^2} \text{ is defined} \} \\ &= \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 \neq 0 \} \\ &= \{ (x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0) \} \end{aligned}$$



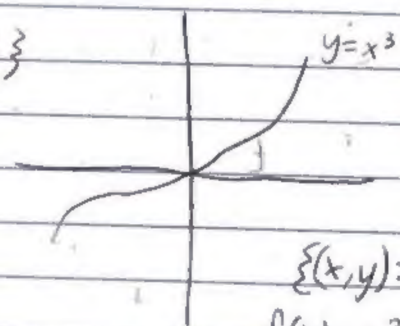
Ex: $f(x,y) = \dfrac{x^2+y^2}{x^2-y^2}$

$$\begin{aligned}\operatorname{dom}(f) &= \left\{(x,y) : \frac{x^2+y^2}{x^2-y^2} \text{ is defined}\right\} \\ &= \{(x,y) \in \mathbb{R}^2 : x^2-y^2 \neq 0\} \\ &= \{(x,y) \in \mathbb{R}^2 : x \neq \pm y\} \\ &= \{(x,y) \in \mathbb{R}^2 : |x| \neq |y|\}\end{aligned}$$



Definition: The <u>graph</u> of a function f is

$\operatorname{graph}(f) = \{(\vec{x}, f(\vec{x})) : \vec{x} \in \operatorname{dom}(f)\}$



$\{(x,y) : y = x^3\}$

$f(x) = x^3$ //

$\{(x, f(x)) : x \in \operatorname{dom}(f)\}$

Ex: What is the shape of $f(x,y) = \sqrt{x^2+y^2+1}$?

Solution: Setting $z = f(x,y)$

$z = \sqrt{x^2+y^2+1}$ ie $z^2 = x^2+y^2+1$ and $z \geq 0$

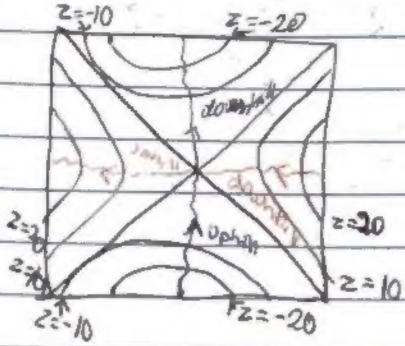
ie $\underbrace{-x^2-y^2+z^2=1}_{\text{two-sheet hyperboloid}}$ and $z \geq 0$

$\therefore$ graph(f) is the upper sheet of a two-sheet hyperboloid

Q: How can we represent ~~[illegible]~~ 2-variable functions in 2-space?

A: Build a contour map (ie. level curves or elevation map)

Ex:



← contour map of a hyperbolic paraboloid

Ex: The unit hyper sphere is $|t| \leq 1$

$$S^3 = \{ (x,y,z,t) \in \mathbb{R}^4 : x^2+y^2+z^2+t^2 = 1 \}$$

The t-level sets look like:

$t=-1$:

starts w/ a point

$t=0$:

sphere gets bigger

$t=-\frac{1}{2}$:

$t=\frac{1}{2}$:

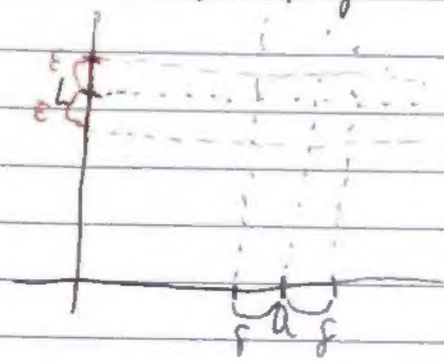
gets smaller

$t=1$:

Section 14.2: Limits and Continuity

In Calc III, the formal definition of a limit is:

Definition: Let f be a multivariable function and let $\vec{a} \in \mathbb{R}^n$ be a limit point of $dom(f)$. The limit as $\vec{x}$ tends to $\vec{a}$ of f is $L \in \mathbb{R}$ when for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $\vec{a} \neq \vec{x} \in dom(f)$ we have $|\vec{x}-\vec{a}| < \delta$ implies $|f(\vec{x}) - L| < \varepsilon$



Notation

$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$

or

$f(\vec{x}) \to L$ as $\vec{x} \to \vec{a}$

Calc III version of "one-sided limits are equal"

Prop (Curves Criterion): Let f be a function and $\vec{a}$ a limit point of its domain. $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$

if and only if for all curves $\vec{r}(t)$ in $dom(f)$ s.t. $\lim_{t \to 0^+} \vec{r}(t) = \vec{a}$ we have $\lim_{t \to 0^+} f(\vec{r}(t)) = L$

Ex: Show $\lim_{(x,y) \to (0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist

Solution: Let $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ and $l_{a,b}(t) = \langle at, bt \rangle$

Note that $\lim_{t \to 0} l_{a,b}(t) = \langle 0, 0 \rangle$

for all $t \neq 0$ we have

$f(l_{a,b}(t)) = \frac{(at)^2-(bt)^2}{(at)^2+(bt)^2} = \frac{(a^2-b^2)t^2}{(a^2+b^2)t^2} = \frac{a^2-b^2}{a^2+b^2}$

$\therefore \lim_{t \to 0} f(l_{0,1}(t)) = \lim_{t \to 0} \left. \frac{a^2-b^2}{a^2+b^2} \right|_{a=0, b=1} = \frac{0-1}{0+1} = -1$, $\lim_{t \to 0} f(l_{1,1}(t)) = 0 \neq -1$

$\therefore$ by curves criterion it does not exist